Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00112-0

J. Appl. Maths Mechs, Vol. 66, No. 5, pp. 805–816, 2002

0021-8928/02/$—see front matter

THE STEADY MOTION OF AN ELASTIC CYLINDER COMPRESSED BY A FIXED SHELL†

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(*Received* 29 *November* 2001)

The steady mixed problem of the motion of a transversely isotropic elastic circular cylinder, compressed by a finite elastic shell, is solved by the method of piecewise-homogeneous solutions [1]. One of the relations of generalized orthogonality obtained for homogeneous solutions is used. Two special cases are considered: (1) a semi-infinite shell is placed on a movable cylinder with a specified negative allowance the edge of the shell is stress-free, and there is no preloading, and (2) a concentrated encircling load acts on the shell. The solution of the problem of a semi-infinite shell and the system of piecewise-homogeneous solutions are constructed in quadratures by the Wiener–Hopf method. (A similar problem was investigated in [2] in a static formulation. Steady mixed contact problems were investigated previously in [3–10].) © 2003 Elsevier Science Ltd. All rights reserved.

1. THE ORTHOGONALITY OF HOMOGENEOUS SOLUTIONS

The orthogonality relations for homogeneous solutions [10] enabled the method of piecewise-homogeneous solutions to be applied to dynamic problems of the theory of elasticity on the steady motions with different velocities of contacting infinite orthotropic cylinders and layers. A generalized orthogonality relation was derived for solving the axisymmetric problem of the steady motion of a transversely isotropic elastic circular cylinder with respect to a fixed thin circular cylindrical shell by the method of piecewise-homogeneous solutions.

In a cylindrical system of coordinates $Or\theta z$ in the case of two circular transversely isotropic cylinders $0 \leqslant r \leqslant R_1$, $-\pi < \theta \leqslant \pi$, $-\infty < z_1 < +\infty$ and $R_1 \leqslant r \leqslant R_2$, $-\pi < \theta \leqslant \pi$, $-\infty < z_2 < +\infty$, moving with respect to this system with velocities w_1 and w_2, $z_1 = z - w_1 t$, $z_2 = z - w_2 t$, where t is the time, the orthogonality relation [10] takes the form

$$\left(\int_0^{R_1} + \int_{R_1}^{R_2}\right)[(\mathbf{u}^m, \mathbf{M}^n) - (\mathbf{M}^m, \mathbf{u}^n)]r\,dr = 0, \quad m \neq -n \tag{1.1}$$

In view of the axial symmetry, the displacement vector $\mathbf{u}^m$ has only two non-zero components u_r^m and u_z^m – the radial and axial displacements, and the components of the vector $M^m = \{M_1^m, M_2^m\}$ have the form

$$M_1^m = \mu \frac{c^2}{c_2^2} \frac{\partial u_z^m}{\partial r} + \left(1 - \frac{c^2}{c_2^2}\right)\tau_{rz}^m, \quad M_2^m = \lambda \frac{c^2}{c_1^2}\left(\frac{\partial u_r^m}{\partial r} + \frac{u_r^m}{r}\right) + \left(1 - \frac{c^2}{c_1^2}\right)\sigma_z^m \tag{1.2}$$

Here τ_{rz}^m, σ_z^m are the shear and normal stresses, the superscript m indicates the fact that the components of the homogeneous solution considered are determined by the mth root p_m ($m = \pm 1, \pm 2, \ldots$) of the dispersion equation, $p_{-m} = -p_m$, $c_1^2 = (\lambda + 2\mu)/\rho$ and $c_2^2 = \mu/\rho$, c_1, c_2 are the velocities of the compression and shear waves, λ and μ are the Lamé coefficients, ρ are the densities, c are the velocities, which, for the first and second cylinders, have the values λ_1, μ_1, ρ_1, w_1 and λ_2, μ_2, ρ_2, w_2 respectively, and (,) is the scalar product.

By reducing the radius R_2 we can change the thickness of a hollow cylinder to a small value h, and take the radius $R = R_1 + h/2$ as the shell thickness. We will represent the second integral in (1.1) in the form

†*Prikl. Mat. Mekh.* Vol. 66, No. 5, pp. 842–854, 2002.

$$\int_{R_1}^{R_2} [(\mathbf{u}^m, \mathbf{M}^n) - (\mathbf{M}^m, \mathbf{u}^n)] r\,dr = I_{mn} - I_{nm} \tag{1.3}$$

$$I_{mn} = \int_{R-h/2}^{R+h/2} (u_r^m M_1^m - M_2^m u_z^n) r\,dr$$

According to expressions (1.2), taking into account the fact that the displacements u_r and u_z in axisymmetric problems depend only on the z coordinate, i.e. they are assumed to be constant over the shell thickness, we have

$$I_{mn} = \left(1 - \frac{w_2^2}{c_2^2}\right) R u_r^m P^n - \left(1 - \frac{w_2^2}{c_1^2}\right) M^m u_z^n - \lambda h \frac{w_2^2}{c_1^2} u_r^m u_z^n \tag{1.4}$$

where M^m and P^m are the moment and shearing force. Substituting expression (1.4) into formula (1.3) and then substituting the expression obtained into (1.1), yield the generalized orthogonality relation with the load

$$\int_0^{R_1} [(\mathbf{u}^m, \mathbf{M}^n) - (\mathbf{M}^m, \mathbf{u}^n)] r\,dr + \left(1 - \frac{w_2^2}{c_2^2}\right) R (u_r^m P^n - P^m u_r^n) +$$

$$+ \left(1 - \frac{w_2^2}{c_1^2}\right) (u_z^m M^n - M^m u_z^n) - \lambda h \frac{w_2^2}{c_1^2} (u_r^m u_z^n - u_z^m u_r^n) = 0, \quad m \neq -n$$

Hence, repeating the transformations of the orthogonality relations (10), based on changing to homogeneous solutions, the components of which are mirror images in the $z = 0$ plane, we obtain

$$\int_0^{R_1} (u_r^m M_1^n - M_2^m u_z^n) r\,dr + I_{mn} = 0, \quad m^2 \neq n^2 \tag{1.5}$$

The quantities u_r, u_z and h are small compared with the other quantities, and hence relation (1.5) can be simplified by neglecting in it the expression containing the product of these quantities. In a cylindrical system of coordinates, connected with the fixed shell ($w_2 = 0$), relation (1.5) takes the form

$$\int_0^{R_1} (u_r^m M_1^n - M_2^m u_z^n) r\,dr + R u_r^m P^n - M^m u_z^n = 0, \quad m^2 \neq n^2 \tag{1.6}$$

2. FORMULATION AND SOLUTION OF THE IN HOMOGENEOUS PROBLEM OF A SEMI-INFINITE SHELL

In a cylindrical system of coordinates $Or\theta z$, we will consider the problem of the contact between a fixed semi-infinite circular shell $r = R$, $-\pi < \theta \leqslant \pi$, $z \geqslant 0$ of constant small thickness h with an elastic cylinder $0 \leqslant r \leqslant 1$, $-\pi < \theta \leqslant \pi$, $-\infty < z_1 < +\infty$, moving with constant sub-Rayleigh velocity c, $z_1 = z - ct$. A load $g(z)$, acting on the shell, a bending moment P_1 and a shearing force P_2, applied to its edge, and also the load $f(z)$ on the free part of the cylinder are axisymmetrical. There is no friction between the shell and the cylinder.

The boundary conditions for the elastic cylinder for $r = 1$ have the form

$$\tau_{rz} = 0, \quad -\infty < z < +\infty \tag{2.1}$$

$$\sigma_r = f(z), \quad z < 0; \quad \eta(z) = \frac{g(z)}{D} + \gamma\left(R - 1 - \frac{h}{2}\right), \quad z > 0 \tag{2.2}$$

$$\frac{\partial^n u_r}{\partial z^n} = -\frac{P_{n-1}}{D}, \quad z = 0, \quad n = 2, 3 \tag{2.3}$$

where

$$\eta(z) \equiv \frac{\partial^4 u_r}{\partial z^4} + \gamma u_r + \frac{\sigma_r}{D}, \quad \gamma = \frac{E_0 h}{DR^2}$$

and D and E_0 are the bending stiffness and the modulus of elasticity of the shell.

The solution of the problem will be sought using Lamé potentials, which, in the axisymmetric case in a moving system of coordinates $O_1 r_1 \theta_1 z_1$, $r_1 = r$, $\theta_1 = \theta$, $z_1 = z - ct$, satisfy the equations

$$\Delta\phi_1 = \frac{1}{c_1^2}\frac{\partial^2\phi_1}{\partial t^2}, \quad \Delta\psi_1 = \frac{1}{c_2^2}\frac{\partial^2\psi_1}{\partial t^2} + \frac{\psi_1}{r_1^2}, \quad \Delta \equiv \frac{\partial^2}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial}{\partial r_1} + \frac{\partial^2}{\partial z_1^2} \tag{2.4}$$

The components of the displacement vector $\mathbf{u}_1$ and the stress tensor are given by the well-known formulae [11]

$$u_r(z_1, r_1, t) = \frac{\partial\phi_1}{\partial r_1} - \frac{\partial\psi_1}{\partial z_1}, \quad u_z(z_1, r_1, t) = \frac{\partial\phi_1}{\partial z_1} + \frac{1}{r_1}\frac{\partial(r_1\psi_1)}{\partial r_1} \tag{2.5}$$

$$\sigma_r = 2\mu\left(\frac{\partial^2\phi_1}{\partial r_1^2} - \frac{\partial^2\psi_1}{\partial z_1 \partial r_1}\right) + \lambda\frac{c^2}{c_1^2}\frac{\partial^2\phi_1}{\partial z_1^2}, \quad \tau_{rz} = 2\mu\left(\frac{\partial^2\phi_1}{\partial r_1 \partial z_1} - \frac{\partial^2\psi_1}{\partial z_1^2}\right) + \mu\frac{c^2}{c_2^2}\frac{\partial^2\psi_1}{\partial z_1^2}$$

where λ and μ are the Lamé coefficients.

Since the solutions are time-independent

$$\mathbf{u}_1(z_1, r_1, t) = \mathbf{u}_1(z_1 + ct, r_1, 0) = \mathbf{u}_1(z, r, 0) \tag{2.6}$$

Suppose

$$\mathbf{u}(z, r) = \mathbf{u}_1(z, r, 0) \tag{2.7}$$

It then follows from relations (2.6) and (2.7) that

$$\frac{\partial\mathbf{u}_1(z_1, r_1, t)}{\partial t} = \frac{\partial\mathbf{u}_1(z, r, 0)}{\partial z}\frac{\partial z}{\partial t} = c\frac{\partial\mathbf{u}(z, r)}{\partial z} \tag{2.8}$$

Similarly

$$\frac{\partial\mathbf{u}_1(z_1, r_1, t)}{\partial r_1} = \frac{\partial\mathbf{u}(z, r)}{\partial r}, \quad \frac{\partial\mathbf{u}_1(z_1, r_1, t)}{\partial z_1} = \frac{\partial\mathbf{u}(z, r)}{\partial z} \tag{2.9}$$

In the system of coordinates $Or\theta z$, connected with the shell, relations (2.5) retain their form, by virtue of Eqs (2.9), while for Eqs (2.4), putting

$$\phi(z, r) = \phi_1(z_1 + ct, r_1, 0), \quad \psi(z, r) = \psi_1(z_1 + ct, r_1, 0)$$

by Eqs (2.8), we have

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + a^2\frac{\partial^2\phi}{\partial z^2} = 0, \quad \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2} + b^2\frac{\partial^2\psi}{\partial z^2} = 0$$

$$a^2 = 1 - c^2/c_1^2, \quad b^2 = 1 - c^2/c_2^2$$

or in terms of Laplace transforms

$$\Phi(p, r) \equiv \int_{-\infty}^{+\infty} \phi(z, r)e^{-pz}dz, \quad \Psi(p, r) \equiv \int_{-\infty}^{+\infty} \psi(z, r)e^{-pz}dz \tag{2.10}$$

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + a^2p^2\Phi = 0, \quad \frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + (b^2p^2 - \frac{1}{r^2})\Psi = 0 \tag{2.11}$$

Taking into account the fact that the required solutions are bounded on the axis of the cylinder, we will take the solutions of Eqs (2.11) in the form of Bessel functions of the first kind of the zeroth and first order, respectively

$$\Phi(p,r) = A(p)J_0(apr), \quad \Psi(p,r) = B(p)J_1(bpr)$$

where $A(p)$ and $B(p)$ are functions determined by the boundary conditions of the problem.

By inverting formulae (2.10) we obtain from relations (2.5)

$$u_r(z,r) = \frac{1}{2\pi i}\int_L [\Phi'(p,r) - p\Psi(p,r)]e^{pz}dp \tag{2.12}$$

$$u_z(z,r) = \frac{1}{2\pi i}\int_L \left[p\Phi(p,r) + \frac{\Psi(p,r)}{r} + \Psi'(p,r)\right]e^{pz}dp$$

Here and henceforth the prime denotes differentiation with respect to r, and L denotes the straight line $\operatorname{Re} p = \varepsilon$.

Substituting the expression

$$\tau_{rz}(z,r) = \frac{\mu}{2\pi i}\int_L p[2\Phi'(p,r) - (1+b^2)p\Psi(p,r)]e^{pz}dp$$

(a consequence of Hooke's law and formulae (2.12)) into boundary conditions (2.1), we obtain, apart from an arbitrary factor $C(p)$

$$A(p) = -\mu(1+b^2)J_1(bp), \quad B(p) = 2a\mu J_1(ap)$$

Formulae (2.12) take the form

$$u_r(z,r) = \frac{1}{2\pi i}\int_L C(p)U_1(p,r)e^{pz}dp, \quad u_z(z,r) = \frac{1}{2\pi i}\int_L C(p)U_2(p,r)e^{pz}dp \tag{2.13}$$

$$U_1(p,r) = a\mu p[(1+b^2)J_1(apr)J_1(bp) - 2J_1(ap)J_1(bpr)]$$

$$U_2(p,r) = \mu p[2abJ_0(bpr)J_1(ap) - (1+b^2)J_0(apr)J_1(bp)] \tag{2.14}$$

We will write the mixed boundary conditions (2.2) in terms of Laplace transforms

$$\sigma^+(p) + \sigma^-(p) = C(p)pN_1(p), \quad \eta^+(p) + \eta^-(p) = C(p)pN_2(p), \quad p \in L \tag{2.15}$$

$$\sigma^+(p) = \int_0^{+\infty} \sigma_r(z,1)e^{-pz}dz, \quad \sigma^-(p) = \int_{-\infty}^0 f(z)e^{-pz}dz$$

$$\eta^+(p) = \int_0^{+\infty}\left[\frac{g(z)}{D} + \gamma\left(R - 1 - \frac{h}{2}\right)\right]e^{-pz}dz, \quad \eta^-(p) = \int_{-\infty}^0 \eta(z)e^{-pz}dz$$

$$N_1(p) = \mu^2\{[(1+b^2)^2J_0(ap)J_1(bp) - 4abJ_0(bp)J_1(ap)]p + 2a(1-b^2)J_1(ap)J_1(bp)\}$$

$$N_2(p) = N_1(p)D^{-1} - a\mu c^2c_2^{-2}(p^4+\gamma)J_1(ap)J_1(bp)$$

The superscripts plus and minus denote that the functions are analytic in the right and left half-planes respectively.

Since $N_j(-p) = N_j(p)$ and $N_j(\bar{p}) = \overline{N_j(p)}$ $(j = 1, 2)$, the zeros of these functions are arranged symmetrically about the coordinate axes of the complex plane, and their imaginary parts are bounded [12]. We will renumber the zeros of the functions $N_1(p)$ and $N_2(p)$, lying in the right half-plane and having non-negative imaginary parts, in the order in which their real parts increase, and denote them by a_k and b_k $(k = 1, 2, \ldots)$. It is well known that the Bessel functions $J_n(ap)$ $(n = 0, 1)$ have only real zeros, which satisfy the asymptotic formulae $p_k = \pi(k + n/2 + 3/4)/a + O(1/k)$. We will denote the positive zeros of the product $J_1(ap)J_1(bp)$, renumbered in the order in which they increase, by d_k. The following formulae then hold for large values of k [12]

$$\operatorname{Re} a_k = \pi(k+k_0)/(a+b) \ (|k_0| \leqslant 2), \quad \operatorname{Re} b_k = d_k + O(k^{-3}) \tag{2.16}$$

Eliminating the function $C(p)$ from Eqs (2.15), we obtain the Wiener–Hopf equation

$$\eta^+(p) + \eta^-(p) = K(p)[\sigma^+(p) + \sigma^-(p)], \quad K(p) = N_2(p)/N_1(p), \quad p \in L \tag{2.17}$$

We will find the solution of the homogeneous equation

$$\eta_0^-(p) = K(p)\sigma_0^+(p), \quad p \in L \tag{2.18}$$

We split Eq. (2.18) into two Riemann problems [13]

$$\eta_j^-(p) = K_j(p)\sigma_j^+(p), \quad p \in L, \quad j = 1,2$$

$$\sigma_0^+(p) = \sigma_1^+(p)\sigma_2^+(p), \quad \eta_0^-(p) = \eta_1^-(p)\eta_2^-(p)$$

Noting that

$$K(0) = \lim_{p \to 0} K(p) = \frac{\Delta}{\delta D}, \quad K(i\beta) \sim A|\beta|^3, \quad \beta \to \pm\infty$$

$$\delta = a^2(1-b^2) + (1+b^2)^2 - 4a^2, \quad \Delta = \delta - \frac{a^2(1-b^2)E_0 h}{2R^2\mu}, \quad A = \frac{a(1-b^2)}{\mu R(c)}$$

($R(c) = 4ab - (1+b^2)^2$ is the Rayleigh function), we put $K_1(p) = Ap^3 \operatorname{ctg}^3 \pi p$. Factorization of the cotangent gives

$$\sigma_1^+(p) = A^{-1/2}\Gamma^3(1/2 + p)/\Gamma^3(1+p)$$

Hence it follows that

$$\sigma_1^+(p) \sim A^{-1/2} p^{-3/2}, \quad p \to \infty$$

The function $K_2(p) = K(p)/K_1(p)$ on the imaginary axis is real, and does not have zeros and poles; in addition

$$K_2(0) = \Delta\pi^3(AD\delta)^{-1}, \quad K_2(i\beta) = 1 + O(e^{-2\pi|\beta|}), \quad \beta \to \pm\infty$$

Since the function $K_2(p)$ on the imaginary axis does not change its sign, its index is equal to zero. Consequently, the solution of the second Riemann problem has the form [13]

$$\sigma_2^+(p) = \exp\left\{-\frac{p}{\pi}\int_0^{+\infty} \frac{\ln K_2(it)}{t^2+p^2}\,dt\right\}, \quad \operatorname{Re} p > 0$$

$$\sigma_2^+(i\beta) = K_2^{-1/2}(i\beta)\exp\left\{\frac{\beta}{\pi i}\int_0^{+\infty} \frac{\ln[K_2(it)/K_2(i\beta)]}{t^2-\beta^2}\,dt\right\}$$

It follows from Eqs (2.17) and (2.18) that

$$\frac{\eta^-(p)}{\eta_0^-(p)} + \frac{\eta^+(p)}{\eta_0^-(p)} = \frac{\sigma^+(p)}{\sigma_0^+(p)} + \frac{\sigma^-(p)}{\sigma_0^+(p)}, \quad p \in L \tag{2.19}$$

Special cases. 1. Suppose a thin shell is placed on a moving cylinder with a negative allowance l. In conditions (2.2) and (2.3) we put

$$f(z) = g(z) = 0, \quad R = 1 + h/2 - l, \quad P_1 = P_2 = 0$$

We split this problem into the fundamental problem

$$\tau_{rz} = 0, \quad \eta(z) = -\gamma l, \quad -\infty < z < +\infty, \quad r = 1$$

the solution of which can be found in an elementary manner

$$u_r^l = \sigma_r^l \frac{1-\nu}{E} r, \quad u_z^l = -2\sigma_r^l \frac{\nu}{E} z, \quad \sigma_r^l = -l\left(\frac{1-\nu}{E} + \frac{R^2}{E_0 h}\right)^{-1} \tag{2.20}$$

(E and ν are the elasticity parameters of the cylinder), and the mixed problem

$$\begin{aligned} &\tau_{rz} = 0, \quad -\infty < z < +\infty, \quad r = 1 \\ &\sigma_r = -\sigma_r^l, \quad z < 0, \quad r = 1; \quad \eta(z) = 0, \quad z > 0 \\ &P_1 = P_2 = 0, \quad z = 0, \quad r = 1 \end{aligned} \tag{2.21}$$

By conditions (2.21)

$$\sigma^-(p) = -\sigma_r^l \int_{-\infty}^{0} e^{-pz} dz = \frac{\sigma_r^l}{p}, \quad \eta^+(p) = 0$$

From relations (2.19), on the basis of the asymptotic estimates

$$\sigma_0^+(p) = O(p^{-3/2}), \quad \frac{\sigma^+(p)}{\sigma_0^+(p)} = O(p), \quad p \to \infty \tag{2.22}$$

obtained from an Abel-type theorem [14] taking into account the fact that the local energy of deformation of the cylinder is finite in the immediate vicinity of the edge of the shell, we obtain from the generalized Liouville theorem

$$\sigma^+(p) + \sigma^-(p) = \frac{\sigma_r^l}{\sigma_0^+(p)} \sigma_0^+(p)(p^{-1} + A_- p + B_-), \quad \sigma_0^+(0) = \left(\frac{D\delta}{\Delta}\right)^{1/2} \tag{2.23}$$

The constants A_- and B_- are found from the equilibrium conditions of half the shell $r = R$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant z < +\infty$. Suppose a moment P_1, whose vector is directed along θ, and a shearing force P_2, directed opposite to the radius r, acts on unit length of the end. Then

$$\begin{aligned} &\int_0^\pi P_2 \sin\theta d\theta = -\int_0^\pi \int_0^{+\infty} \sigma_r(z,1) \sin\theta dz d\theta + \int_0^\pi \eta^+(0) \sin\theta d\theta - 2\int_0^{+\infty} N_\theta(z) dz \\ &\int_0^\pi P_1 \sin\theta d\theta = \int_0^\pi \int_0^{+\infty} \sigma_r(z,1) z \sin\theta dz d\theta + \int_0^\pi \eta^{+*}(0) \sin\theta d\theta + 2\int_0^{+\infty} N_\theta(z) z dz \end{aligned} \tag{2.24}$$

Here

$$\eta^+(0) = \int_0^{+\infty} \eta(z) dz, \quad \eta^{+*}(0) = -\int_0^{+\infty} \eta(z) z dz, \quad N_\theta(z) = \frac{E_0 h}{R} u_r(z,1)$$

The asterisk denotes differentiation with respect to p, and $N_\theta(z)$ is the circumferential force in the shell. Substituting

$$\sigma_r(z,1) = \frac{1}{2\pi i} \int_L [\sigma^+(p) + \sigma^-(p)] e^{pz} dp \tag{2.25}$$

into relations (2.24), changing the order of integration and integrating with respect to θ and z, in view of the homogeneous conditions of the mixed problem we arrive at two equations

$$\frac{\sigma_r^l}{\sigma_0^+(0)}\frac{1}{2\pi i}\int_L \sigma_0^+(p)(p^{-1}+A_-p+B_-)\frac{N_3(p)}{p^j N_1(p)}dp=0,\quad j=1,2$$

where

$$N_3(p)=N_1(p)+\xi J_1(ap)J_1(bp),\quad \xi=-a\mu c^2c_2^{-2}E_0hR^{-1}$$

Closing the contour L from the right by semicircles of large radius, passing between the zeros of the function $N_1(p)$, by the theorem of residues we obtain

$$A_-=\frac{S_3^2-S_2S_4}{S_2^2-S_1S_3},\quad B_-=\frac{S_1S_4-S_2S_3}{S_2^2-S_1S_3}$$

$$S_1=\sum_{m=1}^{+\infty}q_m,\quad S_s=\sum_{m=1}^{+\infty}\frac{q_m}{a_m^{s-1}}+\frac{\zeta}{(s-2)!}\lim_{p\to 0}\frac{d^{s-2}}{dp^{s-2}}\sigma_0^+(p),\quad s=2,3,4$$

$$q_k=\frac{\sigma_0^+(a_k)N_3(a_k)}{N_1'(a_k)},\quad \zeta=1+\frac{a\xi}{2\mu^2\delta}$$

$$\sigma_0^{+*}(0)=-\sigma_0^+(0)\left[6\ln 2+\frac{1}{\pi}\int_0^{+\infty}\frac{\ln[K_2(it)/K_2(0)]}{t^2}dt\right]$$

We substitute expression (2.23) into the left-hand side of the first equation of (2.15) and obtain the function $C(p)$. Then, by formulae (2.13), the integral

$$\frac{1}{2\pi i}\int_{L_-} f(p,l)U_q(p,r)e^{pz}dp,\quad f(p,l)=\frac{\sigma_r^l\eta_0^-(p)}{\sigma_0^+(0)pN_2(p)}\left(p^{-1}+A_-p+B_-\right)\tag{2.26}$$

$$q=1,\ldots,5;\quad u_1=u_r,\quad u_2=u_z,\quad u_3=\sigma_r,\quad u_4=\tau_{rz},\quad u_5=\sigma_z$$

is the solution of mixed problem (2.21) and, added to solution (2.20), forms the required functions $u_{q-}^l(z,r)$. Here L_- is the contour of integration, which coincides with the imaginary axis, with the exception of the point $p=0$, which it circumvents from the right along a semicircle of small radius; the transforms $U_q(p,r)$ are given by formulae (2.14) with $q=1,2$, and the following expressions

$$U_3(p,r)=\mu^2p\left\{\left(1+b^2\right)J_1(bp)\left[\left(1+b^2\right)pJ_0(apr)-2aJ_1(apr)r^{-1}\right]-\right.$$
$$\left.-4aJ_1(ap)\left[bpJ_0(bpr)-J_1(bpr)r^{-1}\right]\right\}$$

$$U_4(p,r)=2a\left(1+b^2\right)\mu^2p^2\left[J_1(apr)J_1(bp)-J_1(ap)J_1(bpr)\right]$$

$$U_5(p,r)=\mu p^2\left\{4ab\mu J_0(bpr)J_1(ap)-\left(1+b^2\right)\left[2\mu+\lambda\left(1-a^2\right)\right]J_0(apr)J_1(bp)\right\}$$

2. Suppose now that a concentrated encircling load $-P$ acts on the shell when $z=l_1$, the edge of the shell is force-free, and there is no load. Under conditions (2.2) and (2.3) we put

$$f(z)=0,\quad g(z)=-P\delta\left(z-l_1\right),\quad R=1+h/2,\quad P_1=P_2=0\tag{2.27}$$

where $\delta(z)$ is the Dirac delta function.

We split this problem into the fundamental problem

$$\tau_{rz}=0,\quad \eta(z)=-PD^{-1}\delta\left(z-l_1\right),\quad -\infty<z<+\infty,\quad r=1$$

the solution of which is found from formulae (2.13) with

$$C(p)=\frac{\eta^{+}(p)+\eta^{-}(p)}{pN_2(p)}=-\frac{P}{DpN_2(p)}\int_{-\infty}^{+\infty}\delta(z-l_1)e^{-pz}dp=-\frac{Pe^{-pl_1}}{DpN_2(p)}$$

and a correcting problem

$$\begin{aligned}&\tau_{rz}=0,\quad -\infty<z<+\infty,\quad r=1\\&\sigma_r=-\sigma_r^{\delta}(z,1),\quad z<0,\quad r=1;\quad \eta(z)=0,\quad z>0\\&\frac{\partial^n u_r}{\partial z^n}=-\frac{P_{n-1}^{\delta}}{D},\quad z=0,\quad r=1,\quad n=2,3\end{aligned}\tag{2.28}$$

where

$$\sigma_r^{\delta}(z,1)=\sum_{k=1}^{+\infty}f_k e^{b_k z},\quad P_{n-1}^{\delta}=\sum_{k=1}^{+\infty}\frac{f_k N_3(b_k)}{b_k^{4-n}N_1(b_k)},\quad f_k=\frac{PN_1(b_k)e^{-b_k l_1}}{DN_2'(b_k)}$$

are the radial stress, the moment ($n = 2$) and the shearing force ($n = 3$), defined by the fundamental problem.

Substituting the Laplace transforms corresponding to conditions (2.28) into Eq. (2.19), by virtue of the asymptotic estimates (2.22) we have

$$\sigma^{+}(p)+\sigma^{-}(p)=\sigma_0^{+}(p)\sum_{k=1}^{+\infty}\frac{f_k}{\sigma_0^{+}(b_k)}\left(\frac{1}{p-b_k}+A_k p+B_k\right)$$

The constants A_k and B_k are calculated in the same way as the coefficients A_- and B_- in the problem of a shell with negative allowance. Substituting the last expression into the right-hand side of (2.25), we obtain from the equilibrium conditions

$$\sum_{k=1}^{+\infty}\frac{f_k}{\sigma_0^{+}(b_k)}\left[A_k S_j+B_k S_{1+j}-S_{4+j}(b_k)\right]=0,\quad j=1,2$$

$$S_{4+j}(b_k)=\sum_{m=1}^{+\infty}\frac{q_m}{a_m^j(b_k-a_m)}+\frac{\zeta\sigma_0^{+}(0)}{b_k^j}+(j-1)\frac{\zeta\sigma_0^{+*}(0)}{b_k}+\frac{\sigma_0^{+}(b_k)N_3(b_k)}{b_k^j N_1(b_k)}$$

Hence

$$A_k=\frac{S_2S_6(b_k)-S_3S_5(b_k)}{S_2^2-S_1S_3},\quad B_k=\frac{S_2S_5(b_k)-S_1S_6(b_k)}{S_2^2-S_1S_3}\tag{2.29}$$

Returning to Eqs (2.15) and (2.18), we obtain

$$C(p)=\frac{\eta_0^{-}(p)}{pN_2(p)}\sum_{k=1}^{+\infty}\frac{f_k}{\sigma_0^{+}(b_k)}\left(\frac{1}{p-b_k}+A_k p+B_k\right)\Bigg],\quad \operatorname{Re}p<0$$

Substituting this expression into (2.13), we obtain, by superposition, the solution of problem (2.1), (2.27)

$$u_{q-}^{\delta}(z,r)=\frac{1}{2\pi i}\int_{L_-}\frac{g(p,l_1)}{pN_2(p)}U_q(p,r)e^{pz}dp,\quad q=1,\ldots,5\tag{2.30}$$

$$g(p,l)=-\frac{P}{D}\left[e^{-pl}-\eta_0^{-}(p)\sum_{k=1}^{+\infty}\frac{N_1(b_k)e^{-b_k l}}{\sigma_0^{+}(b_k)N_2'(b_k)}\left(\frac{1}{p-b_k}+A_k p+B_k\right)\right]$$

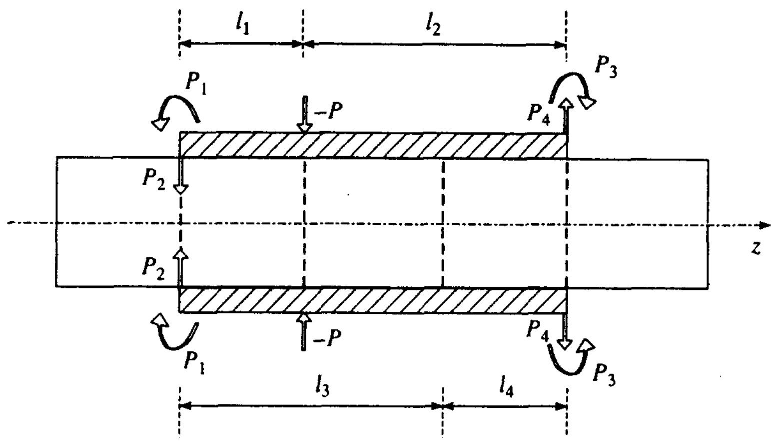

Fig. 1

3. FORMULATION OF THE PROBLEM OF A FINITE SHELL

Consider the problem of the motion, with constant sub-Rayleigh velocity c of a finite circular transversely isotropic cylinder $0 \leqslant r_1 \leqslant 1, -\pi < \theta_1 \leqslant \pi, -\infty < z_1 < +\infty$ relative to a thin shell of finite length and negative allowance l, which compresses it; in addition, a concentrated encircling load $-P$ acts on the shell. In a fixed system of coordinates $O_0r_0\theta_0z_0$, $r_0 = r_1$, $\theta_0 = \theta_1$, $z_0 = z_1 + ct$, the boundary conditions of the problem when $r = 1$ have the form (the zero subscripts of the current coordinates are omitted)

$$\tau_{rz} = 0, \quad -\infty < z < +\infty$$

$$\sigma_r = 0, \quad z < -l_3, \quad z > l_4; \quad \eta(z) = -\gamma l - \frac{P}{D}\delta(z + l_3 - l_1), \quad -l_3 < z < l_4$$

$$\frac{\partial^n u_r}{\partial z^n} = -\frac{P_{n-1}}{D}; \quad z = -l_3; \quad \frac{\partial^n u_r}{\partial z^n} = -\frac{P_{n+1}}{D}, \quad z = l_4; \quad n = 2, 3$$

P_3 is a moment whose vector is directed opposite to θ and P_4 is a shearing force which acts in a radial direction as shown in the figure.

In a semi-infinite cylinder we will construct the solution in the system $Or\theta z$, $r = r_0$, $\theta = \theta_0$, $z = z_0 + l_3$, in the form of the sum of inhomogeneous solutions $u_{q-}^{l}(z, r)$, $u_{q-}^{\delta}(z, r)$, defined by formulae (2.20), (2.26), and (2.30), and a series in piecewise-homogeneous solutions, which satisfy the homogeneous conditions (2.1)–(2.3), with singularities at $z = +\infty$. When $0 \leqslant r_0 \leqslant 1$, $\theta = \theta_0$, $z_0 > 0$ the solution will be sought in coordinates $r = r_0$, $\theta = \theta_0$, $z = z_0 - l_4$ in a similar form for the fundamental condition (2.1) and the mixed conditions

$$\eta(z) = -\gamma l - \frac{P}{D}\delta(z + l_2), \quad z < 0; \quad \sigma_r = 0, \quad z > 0 \tag{3.1}$$

$$\frac{\partial^n u_r}{\partial z^n} = -\frac{P_{n+1}}{D}, \quad z = 0, \quad n = 2, 3 \tag{3.2}$$

with singularities in the piecewise-homogeneous solutions when $z = -\infty$.

We will obtain the coefficients in the series in the piecewise-homogeneous solutions using orthogonality relations (1.6) from the condition of continuity of the solutions in the section $z_0 = 0$, $0 \leqslant r_0 \leqslant 1$, assuming, to fix our ideas, that it is situated to the right of the encircling load: $l_1 < l_3$, $l_2 > l_4$; l_s ($s = 1, \ldots, 4$) are positive constants.

4. SUBSYSTEMS OF PIECEWISE-HOMOGENEOUS SOLUTIONS

We will construct two subsystems of piecewise-homogeneous solutions. According to the description given in Section 3, each element of the first subsystem must satisfy homogeneous conditions (2.1)–(2.3) and have a singularity at $z = +\infty$. These elements are represented in the form of the sum of the solution of the fundamental problem (2.1), $\eta(z) = 0$ $(-\infty < z < +\infty)$, which is found from formulae (2.14) when $p = b_k$, and the solutions of the correcting mixed problem, determined by conditions (2.1)

$$\sigma_r = -\sigma_r^k(z, 1), \quad z < 0, \quad r = 1; \quad \eta(z) = 0, \quad z > 0 \tag{4.1}$$

$$P_1 = -P_1^k, \quad P_2 = -P_2^k, \quad z = 0, \quad r = 1 \tag{4.2}$$

Here

$$\sigma_r^k(z, 1) = C_k b_k N_1(b_k) e^{b_k z}, \quad P_1^k = C_k N_3(b_k) b_k^{-1}, \quad P_2^k = C_k N_3(b_k)$$

is the normal stress, the moment and the shearing force from the kth ($k = 1, 2, \ldots$) homogeneous solution of the fundamental problem, and C_k are arbitrary constants.

Conditions (4.1) lead to the Wiener–Hopf equation

$$\eta^-(p) = K(p)\left[\sigma^+(p) + C_k b_k N_1(b_k)(p - b_k)^{-1}\right], \quad p \in L$$

Repeating the procedure for solving Eq. (2.17) using conditions (4.2) on the end, we obtain

$$C(p) = \frac{C_k b_k N_1(b_k) p^2 \eta_0^-(p)}{\sigma_0^+(b_k) p N_2(p)} \left(\frac{1}{p - b_k} + A_k p + B_k\right)$$

Hence, as a result of correcting the solutions, the elements of the first subsystem of piecewise-homogeneous solutions take the form

$$u_{q-}^k(z, r) = C_k U_q(b_k, r) e^{b_k z} + \\ + \frac{C_k b_k N_1(b_k)}{2\pi i \sigma_0^+(b_k)} \int_{L_-} \frac{\eta_0^-(p)}{p N_2(p)} \left(\frac{1}{p - b_k} + A_k p + B_k\right) U_q(p, r) e^{pz} dp, \quad k = 1, 2, \ldots \tag{4.3}$$

The coefficients A_k and B_k are found from formulae (2.29).

The second subsystem of piecewise-homogeneous solutions is constructed in the same way. Its elements $u_{q+}^k(z, r)$ ($k = -1, -2, \ldots$) satisfy homogeneous boundary conditions (2.1), (3.1) and (3.2) and have a singularity at $z = -\infty$. They are obtained by replacing the contour L_- on the right-hand side of Eq. (4.3) by L_+ (L_+ circumvents the point $p = 0$ from the left), $\eta_0^-(p)$ by $\eta_0^+(p)$, $\sigma_0^+(b_k)$ by $-\sigma_0^-(b_k)$, and A_k and B_k by $-A_k$ and $-B_k$, where

$$\eta_0^+(p) = \eta_0^-(-p), \quad \sigma_0^-(p) = \sigma_0^+(-p), \quad A_{-k} = -A_k, \quad B_{-k} = B_k$$

5. SOLUTION OF THE PROBLEM OF A FINITE SHELL

As in Section 3, the solution of the problem of a finite shell will be sought in the form

$$\begin{aligned} u_{q-}(z, r) &= u_{q-}^l(z, r) + u_{q-}^\delta(z, r) + \sum_{k=1}^{+\infty} u_q^k(z, r) + \sum_{k \in Z_+^C} \bar{u}_q^k(z, r), \quad z < l_3 \\ u_{q+}(z, r) &= u_{q+}^l(z, r) + u_{q+}^\delta(z, r) + \sum_{k=-1}^{-\infty} u_q^k(z, r) + \sum_{k \in Z_-^C} \bar{u}_q^k(z, r), \quad z > -l_4 \end{aligned} \tag{5.1}$$

where $u_{q+}^l(z, r) + u_{q+}^\delta(z, r)$ is the solution of problem (2.1), (3.1) and (3.2). The functions $u_{q+}^l(z, r)$ and $u_{q+}^\delta(z, r)$ differ from $u_{q-}^l(z, r)$ and $u_{q-}^\delta(z, r)$ (see Section 2) by the replacement of $f(p, l)$ and $g(p, l_1)$ and the contour L_- by $f(-p, l)$, $g(-p, l_2)$ and L_+ respectively, and Z_+^C and Z_-^C are sets of numbers of the complex zeros of the function $N_2(p)$, situated in the first and third quadrants of the complex plane, where the dash denotes complex conjugation.

The constants C_k are found from the eight conditions of continuity of the solution when $z_0 = 0$

$$u_{q-}(l_3, r) = u_{q+}(-l_4, r), \quad 0 < r < 1, \quad q = 1, 2, 4, 5 \tag{5.2}$$

$$\frac{\partial^s u_{1-}(l_3,1)}{\partial z^s}=\frac{\partial^s u_{1+}(-l_4,1)}{\partial z^s},\quad s=0,\,1,\,2,\,3 \tag{5.3}$$

by replacing two of the conditions (5.2) (q = 4, 5) by linear combinations

$$u_{6\pm}(z,r)=\mu\frac{c^2}{c_2^2}\frac{\partial u_{2\pm}}{\partial r}+\left(1-\frac{c^2}{c_2^2}\right)u_{4\pm}(z,r)$$

$$u_{7\pm}(z,r)=\lambda\frac{c^2}{c_1^2}\left(\frac{\partial u_{1\pm}}{\partial r}+\frac{u_{1\pm}}{r}\right)+\left(1-\frac{c^2}{c_1^2}\right)u_{5\pm}(z,r)$$

the transforms of which have the form

$$U_6(p,r)=\mu\frac{c^2}{c_2^2}U_2'(p,r)+\left(1-\frac{c^2}{c_2^2}\right)U_4(p,r)$$

$$U_7(p,r)=\lambda\frac{c^2}{c_1^2}\left[U_1'(p,r)+\frac{U_1(p,r)}{r}\right]+\left(1-\frac{c^2}{c_1^2}\right)U_5(p,r)$$

We substitute expressions (5.1) into the new conditions (5.2) and expand the contour integrals in series in residues. Changing the order of summation in the double sums and noting that

$$U_q(-p,r)=(-1)^q U_q(p,r)$$

we obtain

$$\begin{aligned}&\sum_{k=1}^{+\infty}U_q(b_k,r)\left[X_k+(-1)^{q-1}X_{-k}+\Sigma(b_k)\right]+\\&+\sum_{k\in Z_+^C}U_q(\bar b_k,r)\left[\bar X_k+(-1)^{q-1}\bar X_{-k}+\Sigma(\bar b_k)\right]+\delta_{1q}L(r)=0\end{aligned} \tag{5.4}$$

$$X_{\pm k}=C_{\pm k}e^{\pm b_k l_\mp},\quad \Sigma(b_k)=\sum_{n=1}^{+\infty}\left[(-1)^q X_nT_-(b_n,\,-b_k)-X_{-n}T_+(-b_n,\,b_k)+\right.$$

$$\left.+(-1)^q S_-(-b_k,\,b_n)-S_+(b_k,\,-b_n)\right]+\sum_{n\in Z_+^C}\left[(-1)^q\bar X_nT_-(\bar b_n,\,-b_k)-\bar X_{-n}T_+(-\bar b_n,\,b_k)\right]+$$

$$\left.+(-1)^q S_-(-b_k,\,\bar b_n)-S_+(b_k,\,-\bar b_n)\right]+Q_q(b_k)+R_q(b_k)$$

$$L(r)=\frac{r}{\mu}\frac{c^2}{c_2^2}\left(\frac{c^2}{c_1^2}-1\right)\left(\frac{D}{\delta\Delta}\right)^{1/2},\quad Q_q(t)=(-1)^{q-1}\frac{P\left[e^{t(l_1-l_3)}-e^{t(l_4-l_2)}\right]}{DtN_2'(t)}$$

$$R_q(t)=\frac{\sigma_r'\eta_0^+(t)}{\sigma_0^+(0)tN_2'(t)}\left(t^{-1}+A_-t-B_-\right)\left[(-1)^{q-1}e^{-tl_3}+e^{-tl_4}\right]$$

$$S_\pm(t,\tau)=\frac{P\eta_0^\pm(t)N_1(\tau)}{DtN_2'(t)\sigma_0^\mp(\tau)N_2'(\tau)}\left(\pm\frac{1}{\tau-t}+A_{\mp n}t+B_{\mp n}\right)e^{tl_\pm-\tau k_\pm}$$

$$T_\pm(t,\tau)=\frac{tN_1(t)\eta_0^\pm(\tau)}{\sigma_0^\mp(t)\tau N_2'(\tau)}\left(\frac{1}{\tau-t}\mp A_{\mp n}\tau\mp B_{\mp n}\right)e^{(\tau-t)l_\pm}$$

$$q=1,\,2,\,6,\,7;\quad k_-=l_1,\quad k_+=-l_2,\quad l_-=l_3,\quad l_+=-l_4$$

We multiply both sides of the first equation (q = 1) and the fourth equation (q = 7) of system (5.4) by $U_6(b_m,\,r)$ and $-U_2(b_m,\,r)$ respectively, and the second equation (q = 2) and the third equation (q = 6) by $-U_7(b_m,r)$ and $U_1(b_m,r)$ respectively. We add these pairs of equations and integrate the equations obtained with respect to r from 0 to 1. Then, by virtue of the generalized orthogonality relation (1.6)

$$\int_0^1 \left[U_1(b_m,r)U_6(b_n,r) - U_7(b_m,r)U_2(b_n,r)\right]rdr +$$
$$+RU_1(b_m,1)P^n - M^m U_2(b_n,1) = 0, \quad m^2 \neq n^2, \quad R = 1 + h/2$$

and taking conditions (5.3) into account we obtain the normal Poincaré–Koch system with bilateral determinant

$$X_m \pm X_{-m} \mp \sum_{n=1}^{+\infty}\left[X_n T_-(b_n, -b_m) \pm X_{-n} T_+(-b_n, b_m)\right] \mp$$
$$\mp \sum_{n\in Z_+^C}\left[\bar{X}_n T_-(\bar{b}_n, -b_m) \pm \bar{X}_{-n} T_+(-\bar{b}_n, b_m)\right] =$$
$$= h_\pm(b_m) + \sum_{n=1}^{+\infty}\left[S_\mp(\mp b_m, \pm b_n) \pm S_\pm(\pm b_m, \mp b_n)\right] +$$
$$+ \sum_{n\in Z_+^C}\left[S_\mp(\mp b_m, \pm \bar{b}_n) \pm S_\pm(\pm b_m, \mp \bar{b}_n)\right]$$
$$h_+(b_m) = -Q_1(b_m) - R_1(b_m) +$$
$$+\left[DRL(1)b_m^3 U_1(b_m,1) - \int_0^1 L(r)U_6(b_m,r)rdr\right]\times$$
$$\times\left\{\int_0^1\left[U_1(b_m,r)U_6(b_m,r) - U_2(b_m,r)U_7(b_m,r)\right]rdr - D(Rb_m - 1)b_m^2 U_1^2(b_m,1)\right\}^{-1}$$
$$h_-(b_m) = Q_1(b_m) - R_2(b_m); \quad m = 1, 2, \ldots$$

Its matrix elements, according to expressions (2.16), decrease exponentially with the numbers of the rows and columns.

The method described above can be extended in a natural way to the case of any finite number of shells of piecewise-constant stiffness, to the problem of two semi-infinite shells on a cylinder, and also to the problem for periodic systems of shells.

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Translated by R.C.G.